ANALYSIS OF STABILITY IN A SELF-SIMILAR CIRCULAR JET WITH CONSIDERATION OF THE NONPARALLEL EFFECT

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Many experimental and theoretical studies have been devoted to questions dealing with the hydrodynamic stability of free shearing flows. Free flows exhibit a small reserve of stability, and laminar motion is disrupted with even low values for the Reynolds number Re. In actual practice, as a rule, we encounter well-developed turbulent free flows, and the linear analysis of the stability of laminar flows, at first glance, is therefore academic in nature. However, turbulent free flows have successfully been described by the Boussinesq model with a constant turbulent viscosity. The Reynolds numbers Re constructed on the basis of this viscosity are low in value, and the experimentally observed dynamics of such flows involves a sequence of large-scale perturbations developed against a background of small-scale turbulence. Attempts have been made to describe the large-scale motions in turbulent flows by means of the linear theory of stability [1, 2]. Widely accepted in the theoretical study of the stability of free shearing flows is the hypothesis of local parallelism in the average flow, and only in a small number of studies is the effect of nonparallelism taken into consideration. In this case, use is made of a poorly justified procedure which reduces solely to consideration of the transverse component of the average velocity (see, for example, [3]). More acceptable is the expansion of the stream function for the principal and perturbed motions, as well as the expansion of the eigenvalues into a series over the reciprocals of the local Re [4]. Most constructive is the approach to the problem of accounting for nonparallelism that is presented in [5], where the theory of asymptotic expansions over the small parameter is utilized, in this case the reciprocal of the rational power of the global Re. It turns out that for a plane Bickley jet the characteristics of perturbation vary in a non-self-similar manner, i.e., they exhibit various rates of change that are dependent on the longitudinal coordinate. In the present study we examine the example of a flow for which, owing to the nature of its spatial development, the stability can be analyzed by resorting to the hypothesis of perturbation self-similarity. Among such flows we include the self-similar circular Schlichting jet [6]. Analogous relationships with respect to the spatial coordinates are encountered for the averaged characteristics of developed turbulent flow in the jet emanating out of a circular orifice.

In experimental studies of stability in a circular submerged jet it was noted [7] that the flow initially loses stability with respect to axisymmetric perturbations when $\mathrm{Re} \gtrsim 10$. Reynolds [8] cites no values for the critical Re; however, it is his contention that in the range Re = 50-250 both spiral and axisymmetric perturbations are observed within the flow. In [9, 10] we find a thorough study of the nonviscous asymptote of stability parameters for this flow and it has been determined that only the spiral perturbations can be nonstable. This very conclusion has been confirmed in all subsequent studies carried out for finite Re [11-13], from which it follows that in the parallel approximation the flow is unstable only with respect to the spiral perturbations (the azimuthal number m = 1) and loses stability when Re > 38. An attempt was made to explain the absence of agreement between experimental data and theoretical analysis by the difference from self-similarity in the profile of the average velocity and the initial segment of the jet. Indeed, a fuller velocity profile loses stability with respect to perturbations with m = 0, 1, and 2 [13, 14]. However, such a profile exists only at several of the first sections of the jet, while perturbations with m = 0 were observed at considerable distances from the beginning of the jet. All calculations of stability in circular jets have been carried out under the assumption of local parallelism in the original flow.

1. Formulation and Method of Solution for the Problem. The velocity field in the cylindrical coordinate system (X, R, φ) has the components (u, v, w). The following rela-

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tionships serve as a steady solution of the equations of motion and continuity, written in approximation of the boundary layer for a circular submerged jet:

$$U = 1/(1 + r^2)^2, \; V = \varepsilon r(1 - r^2)/(1 + r^2)^2/2 \tag{1}$$

$[r = R/R_0, \; (U, V) = (u, v)/U_0]$. As the characteristic scales we have taken $R_0 = \varepsilon X$, $U_0 = 8\nu/\varepsilon^2 X$, where $\varepsilon = 8/\mathrm{Re}$ and $\mathrm{Re} = (3J/\pi\rho\nu^2)^{1/2}$ is the Reynolds number which is defined by the momentum flux J of the jet, the density ρ of the medium, and by kinematic viscosity ν.

As is usual in linear analysis of stability, the fields of velocity and pressure are presented in the form of an initial steady distribution plus a small perturbation ($\mathbf{u} = \mathbf{U} + \mathbf{u}'$). We will use the affine transformations by means of which we have derived the self-similar relationships (1), for the equations of motion and continuity, linearized with respect to the perturbations:

$$\frac{\partial \mathbf{u}'}{\partial t} + (\mathbf{U}\nabla)\,\mathbf{u}' + (\mathbf{u}'\nabla)\,\mathbf{U} = -\frac{1}{\rho}\nabla p' + \nu\Delta\mathbf{u}', \quad \nabla\mathbf{u}' = 0. \tag{2}$$

We will write the perturbations in the form

$$\begin{Bmatrix}(u', v', w') \\ \mathrm{Re}\, p'\end{Bmatrix} = \begin{Bmatrix}U_0(x)\,[u(r), v(r), w(r)] \\ U_0^2(x)\, q(r)\end{Bmatrix} \exp(i\theta) \tag{3}$$
$$(\partial\theta/\partial X = \alpha_0(x), \; \partial\theta/\partial\varphi = m, \; \partial\theta/\partial t = -\omega_0).$$

Here $x = \varepsilon X$, $r = R/R_0(x)$; ω_0 is the angular frequency of the linear oscillations; m is the azimuthal wave number; $\alpha_0(x)$ is the complex wave number, whose imaginary part characterizes the rate of change in the perturbations, associated with the instability of the latter. Consequently, it is assumed that the amplitude and transverse scale of the small perturbations change slowly with the longitudinal coordinate x, in a manner similar to the change in the average flow, and deviation from self-similar behavior in the oscillations is determined by the imaginary part of their phase. The time derivatives and the derivatives with respect to the longitudinal coordinates in the new variables θ and x, with consideration of the affine transformation of the transverse coordinate, are presented as

$$\partial/\partial t = -\omega_0 \partial/\partial\theta, \; \partial/\partial X = \alpha_0 \partial/\partial\theta + \varepsilon(\partial/\partial x - (\partial \ln R_0/\partial x)\, r\partial/\partial r). \tag{4}$$

It might be assumed that if there exist neutral oscillations of the self-similar form as in (3), their wavelength will also change in self-similar fashion. Moreover, the characteristic scale of the frequency for these neutral oscillations depends on the slowly varying local scales of velocity and length, i.e.,

$$\alpha_0(x) = \alpha/R_0(x), \; \omega_0 = \omega U_0/R_0. \tag{5}$$

One has to ascertain whether or not the frequency is regarded as dependent on the longitudinal coordinates as well as on the parameter, since for given x and Re a specific form of the perturbations is sought, and we have reference here, in particular, to oscillations that are neutral or which grow with some increment. However, if we have to examine the spatial evolution of the small perturbation with a fixed frequency ω_0, the neutral curve $\omega(\mathrm{Re})$ and the scalar similarity (5) define the rate of change in X in which this perturbation will increase for a given Re. Results of this kind are illustrated in [5], where it is demonstrated that for fixed values of the frequency ω_0 and $\mathrm{Re} > \mathrm{Re}_*$ there exists a range of variations in the longitudinal coordinate in which the selected perturbation is unstable. If we substitute a solution of the form of (3) into (2) and if we accomplish the transition to the new variables by means of (4) and (5), the dimensionless equations for the perturbations will be written in the following form:

$$\beta^2 v + q' + i2mw/r^2 - [(rv)'/r]' = \varepsilon\mathrm{Re}[U(rv)' + \varepsilon(rV_1)'/2u - (V_1 v)'/2] + Lv, \tag{6}$$
$$\beta^2 w + imq/r - i2mv/r^2 - [(rw)'/r]' = \varepsilon\mathrm{Re}[U(rw)' - V_1(rw)'/r/2] + Lw,$$
$$\beta^2 u + i\alpha q + \mathrm{Re}U'v - (ru')'/r = \varepsilon[q + (rq)'] +$$

$$+\varepsilon\mathrm{Re}[U(ru)' + (rU)'u - V_1 u'/2] + Lu,$$
$$(rv)'/r + imw/r + i\alpha u - \varepsilon(ru)' = 0.$$

Here $\beta^2 = i\mathrm{Re}(\alpha U - \omega) + \alpha^2 + m^2/r^2$; $V_1 = V/\varepsilon$; $Lf = \varepsilon^2 r(rf)'' + 2\varepsilon(\varepsilon - i\alpha)(rf)' - i\alpha\varepsilon f$. The prime denotes the derivative with respect to r. The boundary conditions for the perturbations

$$\begin{gathered} u, v, w, q \to 0 \text{ as } r \to \infty, \\ u(0) = q(0) = 0,\ m \neq 0,\ v(0) = w(0) = 0,\ m \neq 1, \\ v(0) + iw(0) = 0,\ m = 1, \end{gathered} \tag{7}$$

retain their earlier form [9, 12-14]. If we assume in (6) that $\varepsilon = 0$, we will then derive the well-known system of equations in approximation of parallelism in the initial flow. The nonparallelism parameter ε is a function of the global Re (see above) and changes together with it in appropriate fashion. Solution of the question regarding flow stability involves finding the eigenvalues of α and the eigenfunctions u, v, w, and q of the boundary-value problem (6), (7). The possibility of reducing the stability analysis for a flow that is not one-dimensional to the classical problem involving eigenvalues is governed by the specific nature of the flow in a circular jet, for which, first of all, the transverse scale R_0 coincides with the flow longitudinal coordinate and, second, the local Re is constant throughout the entire flow.

The eigenvalue problem was solved numerically by the method of differential sweeping, with joining in the critical layer [15, 16]. A number of difficulties arise in the solution of the equations for the sweeping coefficients, owing to the presence of singularities at the axis. Equations (6) in the case of r = 0 have a regular singular point in whose vicinity the solution for the system in the form of a series over the powers of r can be found:

$$(v, w) = r^{\gamma}(a_i + b_i r^2 + \ldots),\ (u, q) = r^{\gamma-1}(c_i + d_i r^2 + \ldots),\ i = 1, 2. \tag{8}$$

Substituting (8) into (6) and collecting terms for identical powers of r, we derive the characteristic equation for γ and the recursion relationships which link the constants in expansion (8). The roots of the characteristic equation are equal to (m + 1), (1 − m), −(m + 1), (m − 1), with the first two being multiple. Three linearly independent solutions (u, v, w, q) for $m \neq 0$, bounded when r = 0, have the form

$$\begin{array}{llll} \{-r^m; & i[(\alpha/(m+1) + i\varepsilon)/2]r^{m+1}; & [(\alpha/(m+1) + i\varepsilon)/2]r^{m+1}; & 0\}, \\ \{0; & [m/(m+1)/4]r^{m+1}; & i[(m+2)/(m+1)/4]r^{m+1}; & r^m\}, \\ \{0; & r^{m-1}; & ir^{m-1}; & 0\}. \end{array} \tag{9}$$

Expressions (9) are used to find the sweeping matrix and its first derivative for r = 0, which are necessary in the solution of the equations for the sweeping coefficients. When m = 0 the order of system (6) can be reduced to four. Using the form of the boundary conditions with r = 0, it is not difficult to write the two linearly independent solutions about the axis, but because of their cumbersome nature these are not presented here. In numerical calculations the condition of perturbation attenuation at infinity is replaced by the condition of adhesion at some rather large distance r_0 from the axis. For small α_r the value of r_0 must be increased, since longwave perturbations are extremely sensitive to the conditions at the outer boundary. In order to avoid the effect of the boundary conditions on the calculation results, the integration interval is changed according to the law $r_0 = c_0/\alpha_r$. It was found that when $c_0 = 8$ any further increase in r_0 for a fixed value of α_r has no effect on the results of the calculation of the eigenvalues. The equations for the sweeping coefficients were solved by the Runge–Kutta method with a constant interval. In order to improve calculational accuracy the integration interval was fragmented in the vicinity of the critical layer. The initial eigenvalues of the boundary-value problem (6), (7) for m = 0, 1 were obtained by making the transition (in terms of continuity) from the data presented in [11, 13, 17].

2. Stability of Axisymmetric Perturbations (m = 0). Figure 1, in the (Re, ω) plane, shows the regions in which there exist both stable and unstable small perturbations in the flow being examined here, and where the effects of nonparallelism are taken into considera-

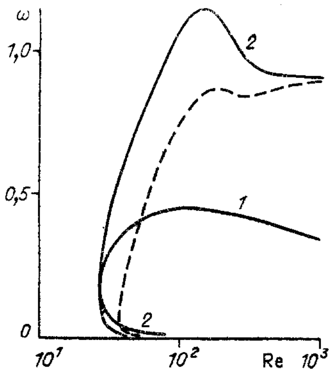

Fig. 1

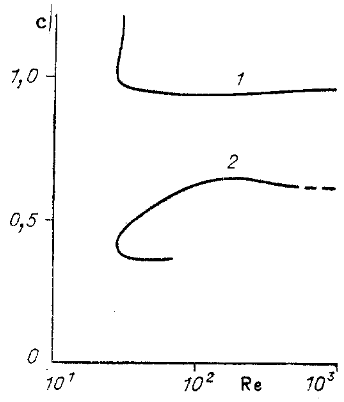

Fig. 2

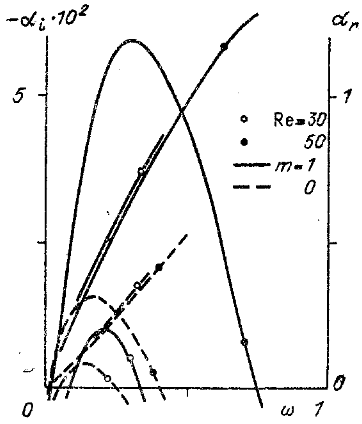

Fig. 3

tion. The neutral curve 1 corresponds to the azimuthal number m = 0, and the nature of the change in this neutral curve is similar to the neutral curves for the flows at the walls, without any bending points in the velocity profile, i.e., with an increase in Re the region of instability degenerates. The critical parameters are $\omega_* = 0.124$, $Re_* = 27.34$, $\alpha_* = 0.125$. The phase velocity $c_* = 0.992$ of the perturbations is virtually equal to the velocity of the flow at the jet axis. The change in c along the neutral curves is shown in Fig. 2 (curve 1), where the upper branch corresponds to $c < 1$. With an increase in Re, $c \to 1$. On the lower branch $c > 1$, which is characteristic of two-dimensional perturbations with $\alpha_r \ll 1$ in problems dealing with the stability of boundary-layer type flows and is associated with the absence of a complete spectrum of eigenvalues [18, 19]. Figure 3 shows α as a function of ω for two values of Re, in excess of the critical value. These results have enabled us to obtain the parameters for the spatial development of harmonic perturbation with a frequency $\omega_0 = \mathbf{U}\Omega/d$ for the chosen Re ($\mathbf{U}$ is the average velocity of the Hagen–Poiseuille flow at the outlet from the nozzle, and d is its diameter). When the Re numbers are constructed on the basis of these dimensions, they coincide with the local $Re = U_0R_0/\nu$, adopted in this study. We will measure the distance along the jet and the perturbation wavelength in nozzle calibers. Using relationships (5) and the definition of $R_0 = x$ as the scale, we derive the expresions $x/d = (\omega/\Omega)^{1/2}$, $\alpha_0 = \alpha(x/d)$, which allows us to present the results in Fig. 3 in the form of Fig. 4. Here we have an illustration of the perturbation increments for three values of the dimensionless frequency Ω as a function of the longitudinal coordinate. The dashed-dotted line corresponds to the maximum values of the increment for Re = 30 (a), $\alpha_{0i} = -0.44 \cdot 10^{-3}/(x/d)$, and 50 (b), $\alpha_{0i} = -1.38 \cdot 10^{-2}/(x/d)$. With a fixed distance from the nozzle, the perturbations with a specific frequency Ω exhibit the greatest growth. The frequency of the most dangerous oscillations $\Omega_* \sim (x/d)^{-2}$ diminishes with increasing distance from the nozzle. Thus, in the process of jet development downstream we find a continuous shifting of the time and space scales for the more dangerous perturbations. For the critical values of the parameters Fig. 5 shows the distributions of the field of pressure and velocity in the perturbations. The perturbed velocity field has the form of toroidal vortices, one following the other, at a phase velocity $c = U_0(x)\omega/\alpha_r$.

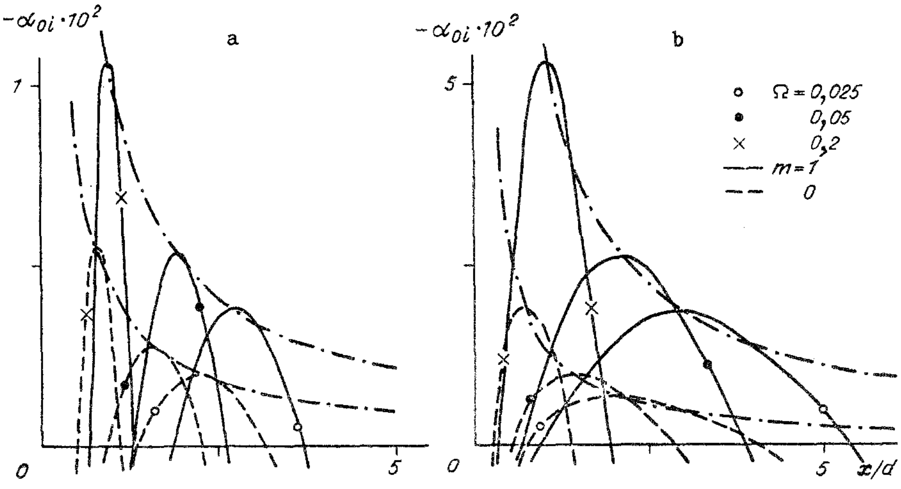

Fig. 4

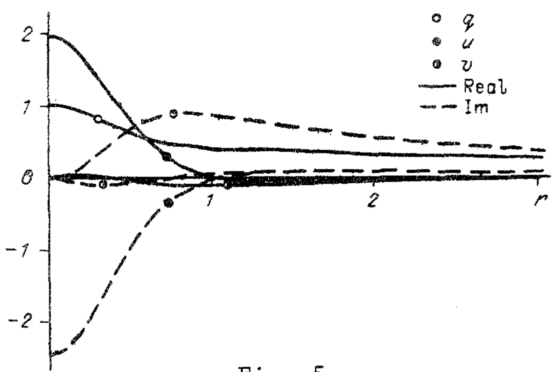

Fig. 5

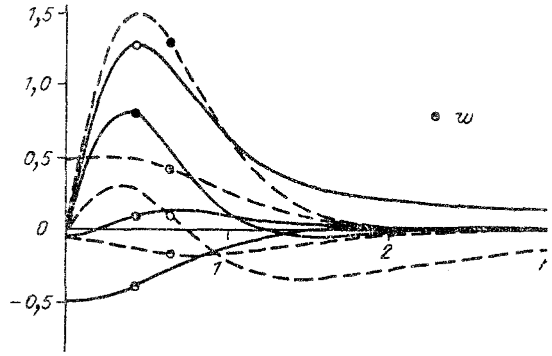

Fig. 6

3. Stability of Spiral Perturbations (m = 1). Line 2 shows the curve for m = 1 in Fig. 1. The dashed line corresponds to the known neutral curve for the parallel approximation ($\omega_* = 0.104$, $Re_* = 37.68$, $\alpha_* = 0.45$). A real jet is less stable, with critical parameters $\omega_* = 0.184$, $Re_* = 27.49$, $\alpha_* = 0.45$. As $Re \to \infty$, both neutral curves exhibit a common asymptote $\omega = 0.91$. The change in the phase velocity along the neutral curve is shown in Fig. 2 (curve 2), and the upper branch corresponds to higher values of c. Consideration of the effects of nonparallelism leads to a significant expansion of the region in which unstable oscillations exist. Some idea as to the relative role of perturbations with m = 0 and 1 can be obtained from Figs. 3 and 4. Although the increments of the axisymmetric perturbations are smaller than for the spiral perturbations, and this difference increases as Re increases, both types of perturbations should be observed in the flow. The maximum values of the increment for m = 1 are determined by the relationships $\alpha_{0i} = -1.02 \cdot 10^{-2}/(x/d)$ for Re = 30 and $\alpha_{0i} = -5.55 \cdot 10^{-2}/(x/d)$ when Re = 50. With increasing Re, the spiral oscillations become predominant. Figure 6 (for the notation see Fig. 5) shows the fields of pressure and velocity in the neutral oscillations for critical values of the parameters. The maximum perturbation in axial velocity is positioned at some distance from the axis of symmetry. The streamlines of the perturbed motion form spirals which are displaced at the phase velocity c along its axis.

4. Discussion of Results and Conclusions. As of the moment we do not have at hand any systematic experimental study into the stability of circular submerged jets. Those few studied carried out in this field are primarily qualitative and descriptive in nature (see, for example, [6, 7]), which makes difficult comparison to theoretical results. Let us take note of the fact derived in the present study, which correlates with experimental observations. In addition to the unstable spiral oscillations in the same range of Re numbers there exist unstable axisymmetric perturbations with a phase velocity close to the velocity of the flow at the axis of the jet. The critical Re numbers are close to those, beginning from which we observe regular perturbations of the two above-cited types. This regularity of perturbations is associated with their extremely weak interaction with the

average flow. According to the proposed self-similarity, it follows from relationships (3) and (5) that the amplitude of the perturbations changes exponentially. In addition to the self-similar portion, the exponent contains the complex number $i\alpha/\varepsilon$, determining the attenuation for increase in perturbations. The exponential law and the not overly large values of the increment α_i/ε for Re $\lesssim$ 100 confirm the slow change in the perturbations along the flow.

In the phenomenological theory of turbulence in free shearing flows, for a description of the average characteristics, the Boussinesq hypothesis regarding turbulent viscosity [6] has been successfully employed. This same hypothesis is used also in analysis of the stability of turbulent flows with respect to limited large-scale perturbations [20]. For a circular jet the turbulent viscosity retains its constant value throughout the entire flow, and the results derived in this study relative to the stability of the laminar circular jet can therefore be extended completely to the case of a turbulent flow.

The calculations that have been carried out show that the role of the effects associated with nonparallelism of the flow in the jet cannot be reduced to a small correction factor for the parallel approximation, but radically alters the overall pattern of stability. This conclusion apparently pertains to analysis of stability and to similar free shearing flows.

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